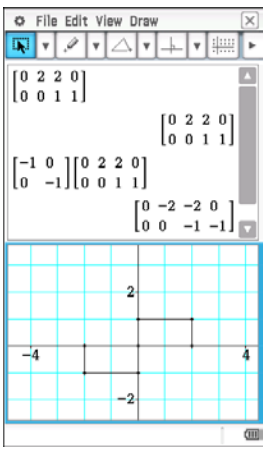
# SADLER MATHEMATICS SPECIALIST UNIT 2

## WORKED SOLUTIONS

### Chapter 11 Transformation matrices

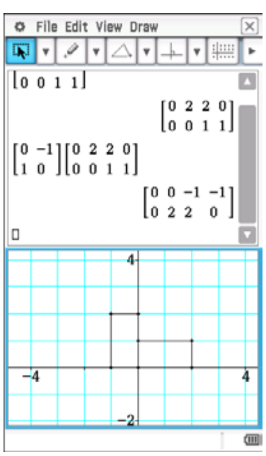
### Exercise 11A

**Question 1**



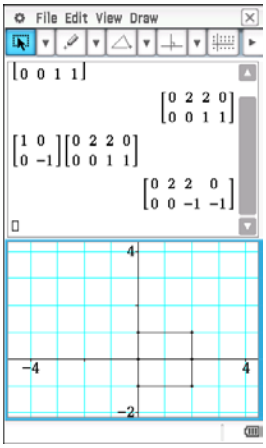

180° rotation about the origin

**Question 2**



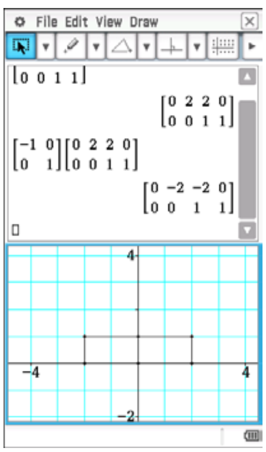

90° anticlockwise rotation about the origin

## Question 3



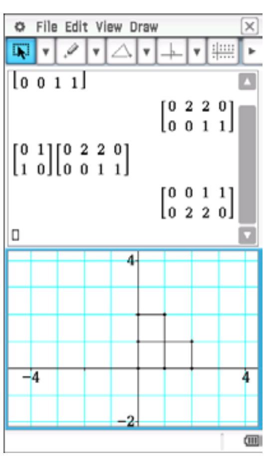

Reflection in *x*-axis

## Question 4



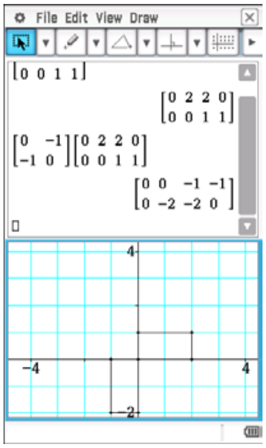

Reflection in *y*-axis

## Question 5



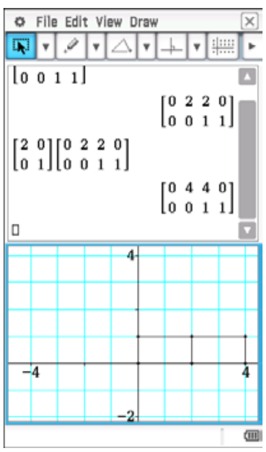

Reflection in the line $y = x$

### Question 6



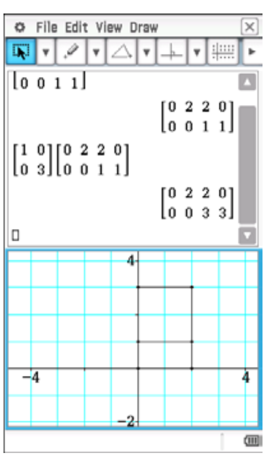

Reflection in the line $y = -x$

### Question 7



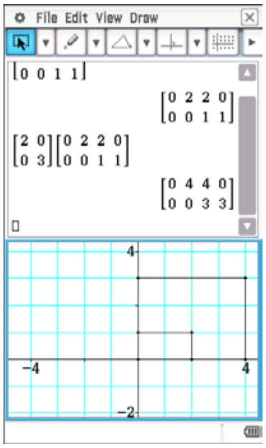

Dilation parallel to $x$-axis, scale factor 2

### Question 8



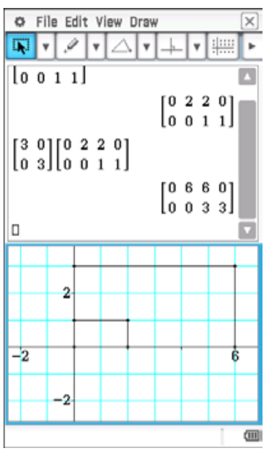

Dilation parallel to $y$-axis, scale factor 3

### Question 9



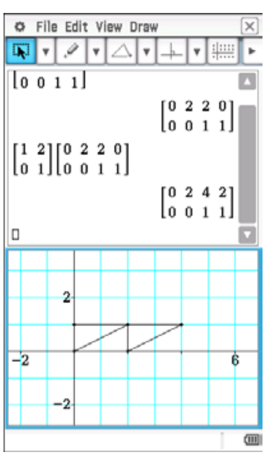
Dilation parallel to the *x*-axis, scale factor 2 and the *y*-axis scale, factor 3

### Question 10



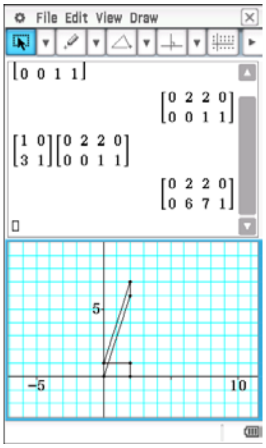
Dilation parallel to the *x*-axis and *y*-axis, scale factor 3

### Question 11



Shear parallel to *x*-axis, scale factor 2

### Question 12




Shear parallel to *y*-axis, scale factor 3

**Question 13**

| Question | Matrix | $\lvert ad - bc \rvert$ | Area of $O'A'B'C'$ | $\frac{\text{Area of O'A'B'C'}}{\text{Area of OABC}}$ |
|---|---|---|---|---|
| 1 | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\lvert 1-0 \rvert = 1$ | 2 | 1 |
| 2 | $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ | $\lvert 0-(-1) \rvert = 1$ | 2 | 1 |
| 3 | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\lvert -1-0 \rvert = 1$ | 2 | 1 |
| 4 | $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\lvert -1-0 \rvert = 1$ | 2 | 1 |
| 5 | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\lvert 0-1 \rvert = 1$ | 2 | 1 |
| 6 | $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ | $\lvert 0-1 \rvert = 1$ | 2 | 1 |
| 7 | $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ | $\lvert 2-0 \rvert = 2$ | 4 | 2 |
| 8 | $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ | $\lvert 3-0 \rvert = 3$ | 6 | 3 |
| 9 | $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ | $\lvert 6-0 \rvert = 6$ | 12 | 6 |
| 10 | $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ | $\lvert 9-0 \rvert = 9$ | 18 | 9 |
| 11 | $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ | $\lvert 1-0 \rvert = 1$ | 1 | 1 |
| 12 | $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ | $\lvert 1-0 \rvert = 1$ | 1 | 1 |

**c** $\dfrac{\text{Area of O'A'B'C'}}{\text{Area of OABC}} = |\,ad - bc\,|$

**Question 1**

**a** Rotation 90° clockwise about the origin

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \therefore A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Rotation of 180°

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \therefore B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rotation of 90° anticlockwise about the origin

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \therefore C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

**b**
$$\begin{aligned} A^2 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= B \end{aligned}$$

**c**
$$\begin{aligned} C^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= B \end{aligned}$$

**d**
$$\begin{aligned} A^3 &= A^2A \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= C \end{aligned}$$

**e** $B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= I$

**f** $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$= C$

**g** $B^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = B$

## Question 2

**a** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represents a reflection in the *x*-axis

**b** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ represents a reflection in the *y*-axis

**c** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ represents a 180° rotation about the origin

**d** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

**e** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

**Question 3**

$$P = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{0-1}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

**Question 4**

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$\text{Determinant}: 2 \times 1 - 0 \times 0 = 3$

**Question 5**

**a** $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & -2 & -5 & -5 \end{bmatrix}$

**b** A′ (0, 1), B′ (1, –2), C′ (3, –5), D′ (2, –5)

## Question 6

Let the coordinates of the triangle be A (a, b), B (c, d) and C (e, f)

$$T^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Students can repeat this working for each individual point or combined the three as shown below.

$$T^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 7 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 7 & 3 & -2 \\ 3 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 4 \\ 3 & 1 & -3 \end{bmatrix}$$

The coordinates are A (1, 3), B (1, 1) and C(4, –3)

**Question 7**

Let the coordinates of the triangle be A (a, b), B (c, d) and C (e, f)

$$T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 1.5 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 2 \end{bmatrix}$$

The co-ordinates are A (1, 3), B (–1, 2) and C (0, 2)

**Question 8**

$$\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$$

**Question 9**

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ will transfrom PQR directly to P″Q″R″

$$\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ will transform P″Q″R″ to PQR

### Question 10

Matrix for the shear parallel to $y$-axis, scale factor 3

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Matrix for 90° rotation clockwise about origin

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Matrix for shear then rotation

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}=\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

### Question 11

Matrix for 90° rotation clockwise about origin

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Matrix for the shear parallel to $y$-axis, scale factor 3

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

Matrix for rotation then shear

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$$

## Question 12

$$\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$a = 2, b = 5, c = 1, d = 3$

## Question 13

**a** $\mathrm{T_2T_1}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$$

**b** $\mathrm{T_3}\left(\mathrm{T_2T_1}\right)$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ -1 & -4 \end{bmatrix}$$

**c** $\mathrm{T_2}^{-1}$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

**d** $\mathrm{T_3T_2}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$\left(\mathrm{T_3T_2}\right)^{-1}$

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

### Question 14

$T_1$: Reflection in $x$-axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$T_2$: Reflection in $y = x$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$T_3$: Rotation 90° CW about origin $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$T_2T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T_3(T_2T_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$T_3 \times T_2 \times T_1 = \mathtt{I}$ (identity matrix)

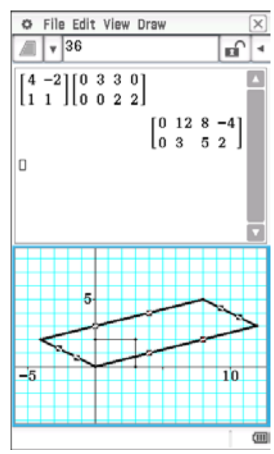
### Question 15

**a** $|T| = |4 \times 1 - 1 \times (-2)|$

$= 6$

Rectangle OABC has an area of 6 square units

Parallelogram O′A′B′C′ has an area of $6 \times 6 = 36$ square units



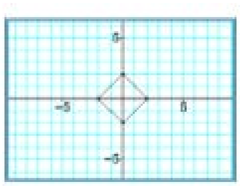

**b** See diagram to right A′ (0, 0), B′ (12, 3), C′ (8, 5), D′ (–4, –2)

**c** See diagram

**d** See diagram

### Question 16

**a**

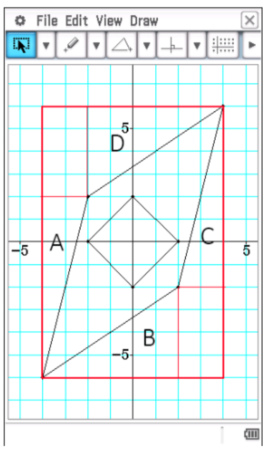


**b** Area = 8 square units

**c** $\det\ M = 5$

Area of $A'B'C'D' = 5 \times 8 = 40$ square units

**d**




Surrounding rectangle area = 96

Triangles A and C: 8

Triangles B and D: 12

Rectangle areas: 16

Area of parallelogram

96 – (16 +24 +16) = 40 square units

### Question 17

All points on the line have the general form $(k, 2k+3)$.

$$\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} k \\ 2k+3 \end{bmatrix}$$

$$= \begin{bmatrix} 2k-(2k+3) \\ -2k+2k+3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$(k, 2k+3)$ transforms to $(3,3)$

## Question 18

All points on the line have the general form $(k, k-1)$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} k \\ k-1 \end{bmatrix}$$
$$= \begin{bmatrix} k \\ 2k+k-1 \end{bmatrix}$$
$$= \begin{bmatrix} k \\ 3k-1 \end{bmatrix}$$

The points on the image line are of the form $x = k$ and $y = 3k-1$.
Eliminating $k$ gives $y = 3x-1$

## Question 19

Let all points be of the form (a, b)

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+3b \\ 3a+9b \end{bmatrix}$$
$$= \begin{bmatrix} a+3b \\ 3(a+3b) \end{bmatrix}$$

All points are of the form $(x, 3x)$ so the equation of the line is $y = 3x$

## Question 20

**a** All points on the line are of the form $(k, 5-3k)$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} k \\ 5-3k \end{bmatrix} = \begin{bmatrix} 6k+2(5-3k) \\ 3k+5-3k \end{bmatrix}$$
$$= \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

All points on the line are transformed to the point (10,5)

**b** $$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6a+2b \\ 3a+b \end{bmatrix}$$

The points on the image line are of the form $x = 6a+2b$ and $y = 3a+b$

The equation of the line is $y = \frac{1}{2}x$

## Question 21

The general form of points on the line $y = m_1 x + p$ is $(k, m_1 k + p)$.

$$\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} k \\ m_1 k + p \end{bmatrix}$$
$$= \begin{bmatrix} 3k \\ 2k + m_1 k + p \end{bmatrix}$$
$$= \begin{bmatrix} 3k \\ (m_1+2)k + p \end{bmatrix}$$

$x = 3k, y = (m_1+2)k + p$

$k = \frac{x}{3}$

$y = (m_1+2)\frac{x}{3} + p$

$y = \frac{(m_1+2)}{3}x + p$ is the equation of the image line

$m_2 = \frac{(m_1+2)}{3}$

Let two lines have gradients $m_A$ and $m_B$, $m_A \times m_B = -1$.

After transformation matrix A, the image of the line with gradient $m_A$ has a gradient of $\frac{m_A + 2}{3}$.

Similarly, the image of the line with gradient $m_B$ has a gradient of $\frac{m_B + 2}{3}$.

The two lines are perpendicular after transformation, $\frac{m_A + 2}{3} \times \frac{m_B + 2}{3} = -1 \Rightarrow (m_A + 2)(m_B + 2) = -9$.

$$
\begin{aligned}
&\text{Given } m_A m_B = -1, m_B = -\frac{1}{m_A} \\
&(m_A + 2)(m_B + 2) = -9 \\
&(m_A + 2)(-\frac{1}{m_A} + 2) = -9. \\
&-1 - \frac{2}{m_A} + 2m_A + 4 = -9 \\
&2m_A - \frac{2}{m_A} + 12 = 0 \\
&m_A^{\,2} - 1 + 6m_A = 0 \\
&m_A^{\,2} + 6m_A - 1 = 0 \\
&(m_A + 3)^2 - 10 = 0 \\
&(m_A + 3)^2 = 10 \\
&m_A + 3 = \pm\sqrt{10} \\
&m_A = -3 \pm \sqrt{10}
\end{aligned}
$$

If $m_A = -3 + \sqrt{10}$,

$$
\begin{aligned}
m_B &= \frac{-1}{-3 + \sqrt{10}} \\
&= -3 - \sqrt{10}
\end{aligned}
$$

If $m_A = -3 - \sqrt{10}$,

$$
\begin{aligned}
m_B &= \frac{-1}{-3 - \sqrt{10}} \\
&= -3 + \sqrt{10}
\end{aligned}
$$

The gradients of the lines before transformation are $-3 - \sqrt{10}$ and $-3 + \sqrt{10}$

## Exercise 11C

### Question 1

**a** $\begin{bmatrix}\cos 30° & -\sin 30° \\ \sin 30° & \cos 30°\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$

**b** $\begin{bmatrix}\cos 45° & -\sin 45° \\ \sin 45° & \cos 45°\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$

**c** $\begin{bmatrix}\cos 60° & -\sin 60° \\ \sin 60° & \cos 60°\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

**d** $\begin{bmatrix}\cos 90° & -\sin 90° \\ \sin 90° & \cos 90°\end{bmatrix} = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$

**e** $\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$

**f** $\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix} = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$

**g** $\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$

**Question 2**

**a** $\begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

**b** $\begin{bmatrix} \cos 120° & \sin 120° \\ \sin 120° & -\cos 120° \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A reflection followed the same reflection will return a shape to its original position.

**Question 3**

Remembering $\cos\theta = \cos(-\theta)$ and $\sin\theta = -\sin(\theta)$, $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

**Question 4**

$$\begin{bmatrix}\cos 120° & \sin 120° \\ \sin 120° & -\cos 120°\end{bmatrix}\begin{bmatrix}\cos 90° & \sin 90° \\ \sin 90° & -\cos 90°\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

**Question 5**

A rotation of angle A followed by angle B can be represented the matrix

$$\begin{bmatrix}\cos B & -\sin B \\ \sin B & \cos B\end{bmatrix}\begin{bmatrix}\cos A & -\sin A \\ \sin A & \cos A\end{bmatrix}$$

$$=\begin{bmatrix}\cos A\cos B-\sin A\sin B & -\sin A\cos B-\sin B\cos A \\ \cos A\sin B+\sin A\cos B & -\sin A\sin B-\cos A\cos B\end{bmatrix}$$

A rotation of angle A followed by angle B is equivalent to a single rotation of angle (A+B).

The matrix for this single rotation is

$$\begin{bmatrix}\cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B)\end{bmatrix}$$

$$\begin{aligned}\cos(A+B) &= \cos A\cos B-\sin A\sin B \\ \sin(A+B) &= \cos A\sin B+\sin A\cos B \\ &= \sin A\cos B+\cos A\sin B\end{aligned}$$

### Question 6

Reflection in $y = m_1 x,\ m_1 = \tan\theta \quad \Rightarrow \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

Reflection in $y = m_2 x,\ m_2 = \tan\phi \quad \Rightarrow \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$

$$\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos 2\phi\cos 2\theta + \sin 2\phi\sin 2\theta & \cos 2\phi\sin 2\theta - \sin 2\phi\cos 2\theta \\ \sin 2\phi\cos 2\theta - \cos 2\phi\sin 2\theta & \sin 2\phi\sin 2\theta + \cos 2\phi\cos 2\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos(2\phi - 2\theta) & \sin(2\theta - 2\phi) \\ \sin(2\phi - 2\theta) & \cos(2\phi - 2\theta) \end{bmatrix}$$
$$= \begin{bmatrix} \cos(2\phi - 2\theta) & \sin(-(2\phi - 2\theta)) \\ \sin(2\phi - 2\theta) & \cos(2\phi - 2\theta) \end{bmatrix}$$

Remembering $\sin(-\theta) = -\sin\theta$ this then becomes

$\begin{bmatrix} \cos(2\phi - 2\theta) & -\sin(2\phi - 2\theta) \\ \sin(2\phi - 2\theta) & \cos(2\phi - 2\theta) \end{bmatrix}$ which represents an anticlockwise rotation of $(2\phi - 2\theta) \Rightarrow \alpha = 2\phi - 2\theta$

$$= 2(\phi - \theta)$$

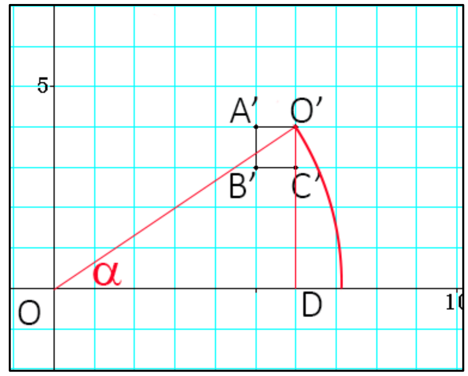
**Question 7**

**a** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

**b** $(OO')^2 = 4^2 + 6^2$

$OO' = \sqrt{52}$

$= 2\sqrt{13}$




In triangle OO′D, $\sin\alpha = \dfrac{4}{2\sqrt{13}} = \dfrac{2}{\sqrt{13}}$

$\cos\alpha = \dfrac{6}{2\sqrt{13}} = \dfrac{3}{\sqrt{13}}$

Matrix required $\begin{bmatrix} \dfrac{3}{\sqrt{13}} & \dfrac{2}{\sqrt{13}} \\ -\dfrac{2}{\sqrt{13}} & \dfrac{3}{\sqrt{13}} \end{bmatrix} = \dfrac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$

**c**

$$\frac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 6 & 5 & 5 & 6 \\ 4 & 4 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2\sqrt{13} & \dfrac{23\sqrt{13}}{13} & \dfrac{21\sqrt{13}}{13} & \dfrac{23\sqrt{13}}{13} \\ 0 & \dfrac{2\sqrt{13}}{13} & \dfrac{-\sqrt{13}}{13} & \dfrac{-3\sqrt{13}}{13} \end{bmatrix}$$

$O''(2\sqrt{13}, 0)$, $A''\left(\dfrac{23\sqrt{13}}{13}, \dfrac{2\sqrt{13}}{13}\right)$, $B''\left(\dfrac{21\sqrt{13}}{13}, \dfrac{-\sqrt{13}}{13}\right)$, $C''\left(\dfrac{23\sqrt{13}}{13}, \dfrac{-3\sqrt{13}}{13}\right)$

## Question 1

$$\begin{aligned}\text{LHS} &= \cos^4\theta - \sin^4\theta \\ &= (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) \\ &= \cos^2\theta - (1-\cos^2\theta) \\ &= 2\cos^2\theta - 1 \\ &= \cos 2\theta \\ &= \text{RHS}\end{aligned}$$

## Question 2

$$2\cos^2 x + \sin x - 2\cos 2x = 0$$
$$2(1-\sin^2 x) + \sin x - 2(1-2\sin^2 x) = 0$$
$$2 - 2\sin^2 x + \sin x - 2 + 4\sin^2 x = 0$$
$$2\sin^2 x + \sin x = 0$$
$$\sin x(2\sin x + 1) = 0$$
$$\sin x = 0 \quad \text{or} \quad 2\sin x + 1 = 0$$
$$x = 0, \pi, 2\pi \qquad x = \frac{7\pi}{6}, \frac{11\pi}{6}$$
$$x = 0, \frac{7\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi$$

## Question 3

$$\begin{aligned}\cos(2\theta + \theta) &= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta + 2\sin\theta\cos\theta\sin\theta \\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

$$a = 4, b = 0, c = -3, d = 0$$

**Question 4**

**a** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $|\det| = |0\times 0 - 1\times(-1)| = 1$

**b** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $|\det| = |(-1)\times(-1) - 0\times 0| = 1$

**c** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $|\det| = |1\times(-1) - 0\times 0| = 1$

**d** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $|\det| = |0\times 0 - 1\times 1| = 1$

**e** $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ $|\det| = |1\times 1 - 0\times 4| = 1$

**f** $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $|\det| = |1\times 1 - 0\times 3| = 1$

**Question 5**

$A_{2\times 3}$ $B_{1\times 2}$ $C_{3\times 4}$

$B_{1\times 2}A_{2\times 3}C_{3\times 4}$

Order of multiplication: BAC

$$\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 3 \\ 3 & 1 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 & 2 & 4 \end{bmatrix}$$

## Question 6

**a** Cannot be determined – matrices are not the same size

**b** $\begin{bmatrix} 3 & 1 \\ 0 & 5 \end{bmatrix}$

**c** Cannot be determined – number of columns in matrix 1 does not equal the number of rows in matrix 2.

**d** $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 1 & -3 & -2 \end{bmatrix}$

**e** $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 6 \end{bmatrix}$

**f** $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 10 \end{bmatrix}$

**g** $\text{BA}_{2\times 3}$ $\text{C}_{2\times 2}$

BA + C cannot be determined – matrices are not the same size

## Question 7

$2x^2 - (-4) = 0$

$2x^2 + 4 = 0$

$2x^2 = -4$

$x^2 = -2$

No real $x$ as a solution to this equation

**Question 8**

$$A^2 = \begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} k^2 - 12 & 4k - 4 \\ -3k + 3 & -11 \end{bmatrix}$$

$$\begin{bmatrix} k^2 - 12 & 4k - 4 \\ -3k + 3 & -11 \end{bmatrix} + \begin{bmatrix} k & 4 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 0 & p \\ q & -12 \end{bmatrix}$$

$$\begin{bmatrix} k^2 + k - 12 & 4k \\ -3k & -12 \end{bmatrix} = \begin{bmatrix} 0 & p \\ q & -12 \end{bmatrix}$$

$k^2 + k - 12 = 0$

$(k + 4)(k - 3) = 0$

$k = -4, 3$

$p = 4k$

$p > 0$ so $k = 3$

$p = 4 \times 3$

$= 12$

$q = -3k$

$= -3 \times 3$

$= -9$

**Question 9**

**a** $$AB = \begin{bmatrix} 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 2 + (-2) \times 0 + 2 \times (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \end{bmatrix}$$

**b** $$BA = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 4 \\ 0 & 0 & 0 \\ -1 & 2 & -2 \end{bmatrix}$$

## Question 10

$45 - x^2 = 4x$

$y^2 - y = 4 - y$

$y^2 + 5y = -6$

$6x - 5 = x^2$

$$45 - x^2 = 4x \qquad 6x - 5 = x^2$$
$$x^2 - 4x - 45 = 0 \qquad x^2 - 6x + 5 = 0$$
$$(x+9)(x-5) = 0 \qquad (x-5)(x-1) = 0$$
$$x = -9, 5 \qquad x = 1, 5$$
$$\Rightarrow x = 5$$

$$y^2 = 4 \qquad y^2 + 5y + 6 = 0$$
$$y = \pm 2 \qquad (y+2)(y+3) = 0$$
$$y = -2, -3$$
$$\Rightarrow y = -2$$

## Question 11

$$\text{AB} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3x & 3y \\ 0 & z \end{bmatrix}$$

$$\text{BA} = \begin{bmatrix} x & y \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3x & y \\ 0 & z \end{bmatrix}$$

$3y = y \Rightarrow y = 0$

No other restrictions necessary as

$3x = 3x$ and $z = z$

**Question 12**

$$M^{-1} = \frac{1}{2}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix}$$

$$M^{-1}M^{-1} = \frac{1}{4}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} a & 1 \\ -2 & 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} a^2-2 & a \\ -2a & -2 \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} a^2-2 & a \\ -2a & -2 \end{bmatrix} = \begin{bmatrix} b & 1 \\ c & d \end{bmatrix}$$

$$\frac{1}{4}a = 1 \Rightarrow a = 4$$

$$b = \frac{1}{4}(4^2 - 2) = 3.5$$

$$c = \frac{1}{4}(-2 \times 4) = -2$$

$$d = \frac{1}{4} \times (-2) = 0.5$$

**Question 13**

**a** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

**b** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

**Question 14**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \end{aligned}$$

**Question 15**

$$\begin{aligned} \tan\left(2(x-1.5)\right) &= 2.3 \\ 2(x-1.5) &= 1.16 + n\pi,\ \ n \in \mathbb{Z} \\ x - 1.5 &= 0.58 + \frac{n\pi}{2},\ \ n \in \mathbb{Z} \\ x &= 2.08 + \frac{n\pi}{2},\ \ n \in \mathbb{Z} \end{aligned}$$